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A Duality Theory for a Class of Generalized Fractional Programs

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Abstract. In generalized fractional programming, one seeks to minimize the maximum of a finite number of ratios. Such programs are, in general, nonconvex and consequently are difficult to solve. Here, we consider a particular case in which the ratio is the quotient of a quadratic form and a positive concave function. The dual of such a problem is constructed and a numerical example is given.

Key words: Fractional program, Multi-ratios, Conjugate duality, Convexity.

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1. Introduction

We consider a mathematical program of the form

(P): $\inf\{\max\{Q_i(x)/f_i(x) : i \in I\} : Cx \leq b\}$

with C some $m \times n$-matrix $b \in R^m$ a given vector, $Q_i(x) = 1/2\, x^T A_i x$, A_i positive definite and $f_i : R^n \to R$ a strictly positive concave function. Moreover, I represents a finite index set.

Program P is generally referred to as a generalized fractional program and is generally nonconvex. It has been studied extensively in the case for which $Q_i(x)$, $i \in I$ are arbitrary positive convex functions. For a recent review, see Schaible (1995) in which a full discussion of theory, duality, applications and algorithms has been included. An example of the above formulation in location-allocation with congestion effects has been given by Barros (1995).

In this paper we focus on the numerator functions being a quadratic form and on the concept of duality. For arbitrary numerator functions, there are a variety of approaches to finding a corresponding dual program involving quasiconcave duality (Crouzeix et al., 1983), convex analysis (Jagannathan et al., 1983), and conjugate functions (Scott et al., 1989). Alternative approaches have been given by Bector and Suneja (1988), Boncompte and Martinez-Legaz (1991), and Chandra et al. (1986). Jagannathan and Schaible (1983) show that a symmetric duality theory can be established where the dual involves infinitely many ratios; with conjugate duals and $f_i(x)$ affine, it was shown by Scott and Jefferson (1989) that the dual could be both symmetric and have finitely many ratios. Recent computational results based on the dual program are given by Barros et al. (1996a, b). In this paper, we show that a particularly simple dual involving a convex program with one convex constraint results. The key to this simplicity comes from the fact that the numerator being a quadratic form has a convex square root and hence the ratio is a convex function (Bector, 1968). Consequently, Program (P) is a convex program which is not true for arbitrary $Q_i(x)$. In the particular case that $f_i(\cdot)$, $i \in I$ are affine, the dual is a linear program with one quadratic constraint.

Section 2 derives the dual program to (P) as well as the optimality conditions that relate the primal and dual programs. Section 3 completes the paper with a numerical example.

2. Derivation of the dual program

Clearly the optimization problem (P) is equivalent to

$$\inf\{t : t_i^{-1}Q_i(x) - f_i(x) \le 0, i \in I, t - t_i = 0, i \in I,\\ Cx \le b, t > 0, t_i > 0, i \in I\}$$

Since Q_i is a quadratic convex function (actually $Q_i(x) = 1/2\|A_i^{1/2}x\|_2^2$ with $\|\cdot\|_2$ denoting the Euclidean norm) it follows by Theorem 5.16 of Avriel et al. (1988) that the function $(x, t_i) \to t_i^{-1}Q_i(x)$ is convex on $R^n \times (0, \infty)$ and so (P) is a convex programming problem. Penalizing the restrictions $Cx - b \le 0$ by the Lagrangian multiplier $u \in R_+^m$, the restrictions $t_i^{-1}Q_i(x) - f_i(x) \le 0$, $i \in I$ by the Lagrangian multiplier $\lambda_i \ge 0$ and the restriction $t_i - t = 0$, $i \in I$, by the Lagrangian multiplier $\mu_i \in R$ we obtain for $\lambda := ((\lambda_i)_{i\in I})$ and $\mu := ((\mu_i)_{i\in I})$ the Lagrangian function $\theta(\lambda, \mu, u)$ given by

$$\begin{aligned}
&\inf\left\{\left(1 - \sum_{i\in I}\mu_i\right)t + \sum_{i\in I}(\lambda_i t_i^{-1}Q_i(x) + \mu_i t_i)\right.\\
&\qquad\left. - \sum_{i\in I}\lambda_i f_i(x) + u^T(Cx - b) : x \in R^n, t > 0, t_i > 0, i \in I\right\}\\
&= \inf\left\{\left(1 - \sum_{i\in I}\mu_i\right)t : t > 0\right\} + \inf\left\{\sum_{i\in I}(\lambda_i t_i^{-1}Q_i(x) + \mu_i t_i)\right.\\
&\qquad\left. - \sum_{i\in I}\lambda_i f_i(x) + u^T Cx : x \in R^n, t_i > 0\right\} - u^T b\\
&= \inf\left\{\left(1 - \sum_{i\in I}\mu_i\right)t : t > 0\right\} + \inf_x\left\{\sum_{i\in I}\inf_{t_i>0}\{\lambda_i t_i^{-1}Q_i(x) + \mu_i t_i\}\right.\\
&\qquad\left. - \sum_{i\in I}\lambda_i f_i(x) + u^T Cx\right\} - u^T b
\end{aligned}$$

It is easy to verify by the above expression for the Lagrangian function that the effective domain of θ is contained in

$$D : \left\{(\lambda, \mu, u) : \lambda \geq 0, u \geq 0, \mu_i \geq 0, \sum_{i \in I} \mu_i \leq 1\right\}$$

To compute the Lagrangian function for $(\lambda, \mu, u) \in D$ we observe the following. By elementary calculus it follows for every $x \in R^n$ that

$$\inf\{\lambda_i t_i^{-1} Q_i(x) + \mu_i t_i : t_i > 0\} = 2(\lambda_i \mu_i Q_i(x))^{1/2}$$

and so for $(\lambda, \mu, u) \in D$ we obtain

$$\theta(\lambda, \mu, u) = \inf\left\{2\left(\sum_{i \in I} (\lambda_i \mu_i Q_i(x))^{1/2} - \sum_{i \in I} \lambda_i f_i(x) + u^T Cx : x \in R^n\right\} - u^T b\right.$$

Since $\lambda_i \mu_i \geq 0$ and $Q_i(x) = 1/2\, x^T A_i x$ it follows that the function $\gamma_i : R^n \to R$ given by $\gamma_i(x) := 2(\lambda_i \mu_i Q_i(x))^{1/2}$ is a finite nonnegative positively homogeneous convex function or equivalently a finite gauge. Denoting now by h^* the conjugate function of the function h, a shorthand notation for $\theta(\lambda, \mu, u)$ is given by $-(\sum_{i \in I} \gamma_i + \sum_{i \in I} (-\lambda_i f_i))^*(-C^T u) - u^T b$ and this implies by the previous observations and Theorem 6.4. of Rockafellar (1970) that

$$\theta(\lambda, \mu, u) = -\min\left\{\sum_{i \in I} \gamma_i^*(y_i^*) + \sum_{i \in I} (-\lambda_i f_i)^*(x_i^*) : \sum_{i \in I} x_i^* + \sum_{i \in I} y_i^* = -C^T u\right\} - u^T b$$

Since γ_i is a finite gauge it is well-known (see Theorem 13.2 and Corollary 13.2.1 of Rockafellar (1970)) that $\gamma_i^*(y_i^*) = \delta_c(y_i^*)$ with C:=$\partial\gamma_i(0)$, the subgradient set of γ_i at 0, and $\delta_C\ (\cdot)$ the indicator function of the set C. Hence for (λ, μ, u)$\in D$ the Lagrangian function is simplified to

$$\theta(\lambda, \mu, u) = -\min\left\{\sum_{i \in I} (-\lambda_i f_i)^*(x_i^*) : y_i^* \in \partial\gamma_i(0), i \in I, \sum_{i \in I} x_i^* + \sum_{i \in I} y_i^* = -C^T u\right\} - u^T b$$

Due to $\gamma_i(x) = (2\lambda_i \mu_i)^{\frac{1}{2}} \|A_i^{\frac{1}{2}} x\|_2$ it follows by Example 3.2 of Chapter 6 of Hiriart-Urruty and Lemaréchal (1993) that

$$\partial\gamma_i(0) = A_i^{\frac{1}{2}} B\left(0, (2\lambda_i, \mu_i)^{\frac{1}{2}}\right)$$

with $B(0,r) := \{x \in R^n : \|x\|_2 \le r\}$. This implies $y_i^* \in \partial\gamma_i(0)$ if and only if $y_i^{*T} A_i^{-1} y_i^* \le 2\lambda_i\mu_i$ and so for $\nu(P)$ finite and (LD) given by

$$\sup\{\theta(\lambda,\mu,u) : \lambda \ge 0, u \ge 0; \mu \in R^{|I|}\} \tag{LD}$$

it follows by the strong Lagrangian duality theorem (see Theorem 28.2 of Rockafellar (1970)) that

$$\begin{aligned}\nu(P) &= \nu(LD) = -\min\{-\theta(\lambda,\mu,u) : (\lambda,\mu,u) \in D\}\\ &= -\min\Bigg\{u^T b + \sum_{i\in I}(-\lambda_i f_i)^*(x_i^*) : \sum_{i\in I} y_i^* + \sum_{i\in I} x_i^*\\ &= -C^T u, y_i^{*T} A_i^{-1} y_i^* \le 2\lambda_i\mu_i, i \in I, \lambda \ge 0, \mu \ge 0, u \ge 0, \sum_{i\in I}\mu_i \le 1\Bigg\}\end{aligned}$$

Since the feasible region of the above problem is contained in (use convention $0/0 := 0$ and $\alpha/0 = \infty$ for $\alpha > 0$).

$$F := \left\{\sum_{i\in I} y_i^* + \sum_{i\in I} x_i^* = -C^T u, \sum_{i\in I} \lambda_i^{-1} y_i^* \le 2, \lambda \ge 0, u \ge 0\right\},$$

it follows that $\nu(LD) \le -\min\{u^T n + \sum_{i\in I}(-\lambda_i f_i)^*(x_i^*) : ((y_i^*)_{i\in I}, (x_i^*)_{i\in I}, u, \lambda) \in F\}$. Moreover, any feasible solution of the above problem can be transformed to a feasible solution of the dual problem (LD) (take $\mu_i = 1/2\,\lambda_i^{-1} y_i^{*T} A_i^{-1} y_i^*$) and this finally implies that

$$\nu(P) = -\min\left\{u^T b + \sum_{i\in I}(-\lambda_i f_i)^*(x_i^*) : ((y_i^*)_{i\in I}, (x_i^*)_{i\in I}, u, \lambda) \in F\right\}$$

We note that the dual program has a convex objective, one convex constraint, and a set of linear constraints. Nonlinear programming algorithms generally prefer fewer explicit nonlinear constraints as in the dual. The dual variables u are Lagrangian multipliers on the primal polyhedral constraints and will satisfy a complementary slackness condition. The dual is somewhat complicated by the multipliers $\lambda_i, i \in I$. For $\lambda_i > 0$ at optimality, the corresponding term 'i' in the primal objective will determine the optimal primal value. However, since many of these terms will not contribute to the objective value, their corresponding λ_i will be zero. The most straightforward way to handle these positively homogeneous convex functions in a computational algorithm is to add a constraint $\lambda \ge \varepsilon, \varepsilon > 0$ so as to identify the finite and zero components of λ. A numerical example will illustrate the process.

The primal and dual variables are related at optimality in the following way:

$$x_i^*/\lambda_i \varepsilon \partial f_i(x) \quad \lambda_i > 0 \tag{1}$$

$$y_i^*/\lambda_i = A^T x/t \quad \lambda_i > 0 \tag{2}$$

$$u^T(Cx - b) = 0 \tag{3}$$

$$\operatorname*{Max}_i\{Q_i(x)/f_i(x)\} + b^T u + \sum_{i\in I}(-\lambda_i f_i)^*(x_i^*) = 0 \tag{4}$$

Equation (4) represents the fact that the primal and dual objectives sum to zero at optimality. Equation (3) is the usual complementary slackness condition and (1) and (2) relate the dual variables x_i^*, y_i^* and λ to the primal variable x. Collectively, these results allow the primal/dual solution to be constructed from the dual/primal solution.

We now particularize the results to $f(\cdot)$, an affine function and to the single ratio case.

Case 1. Single Ratio Case $N = 1$

In this case, $\lambda(= \lambda_1)$ is strictly positive, which simplifies, in turn, the positively homogeneous convex function in the constraints. The dual program is

$$\begin{array}{ll} \text{min} & b^T u + (-\lambda f)^*(x^*) \\ \text{s.t.} & x^* + y^* = -C^T u \\ & y^{*T} A^{-1} y^* \le 2\lambda \\ & u \ge 0, \lambda > 0 \end{array}$$

as given previously by Scott and Jefferson (1996).

Case 2. $f(\cdot)$ is an affine function.

Here, $f_i(x_i) = a_i^T x^i + c_i$ and $(-\lambda f)^*(x_i^*) = c_i$ with $x_i = -a_i$. Consequently, the dual program is

$$\begin{array}{ll} \text{min} & b^T + \sum_{i\in I} c_i\lambda_i \\ \text{s.t.} & \sum_{i\in I}(-a_i\lambda_i + x_i^*) = -C^T u \\ & \sum_{i\in I} \lambda_i^{-1} x_i^{*T} A_i^{-1} x_i^* \le 2 \\ & u \ge 0, \lambda \ge 0 \end{array}$$

In this case, we have a linear program with one convex constraint. A specialized algorithm could be developed, along the lines of Martein and Schaible (1989), for linear programs with one quadratic constraint or, alternatively, a general purpose algorithm could be used.

3. A numerical example

Consider the following generalized fractional program:

$$\text{min} \qquad \text{max}\left\{\frac{x^2 + 2y^2}{2x + 3y + 4}, \frac{3x^2 + 0.5y^2}{x + 3y + 2}, \frac{x^2 + 6y^2}{x + 0.5y + 1}\right\}$$

$$x, y$$

$$\text{s.t.} \quad 12x + 3y \geq 6 \tag{5}$$

$$x + y \leq 2 \tag{6}$$

Using the prescription in Case 2 of Section 3, the dual program is

$$\begin{array}{ll}
\min & -6u_1 + 2u_2 + 4\lambda_1 + 2\lambda_1 + \lambda_3 \\
\text{s.t.} & -2\lambda_1 - \lambda_2 - \lambda_3 + x_{11} + x_{21} + x_{31} = 12u_1 - u_2 \\
& -3\lambda_1 - 3\lambda_2 - 0.5\lambda_3 + x_{12}^* + x_{22}^* + x_{32}^* = 3u_1 - u_2 \\
& \lambda_1^{-1}\left(\frac{1}{4}\,x_{11}^* + \frac{1}{8}\,x_{12}^{*2}\right) + \lambda_2^{-1}\left(\frac{1}{12}\,x_{21}^{*2} + \frac{1}{2}\,x_{22}^{*2}\right) \\
& +\lambda_3^{-1}\left(\frac{1}{4}\,x_{31}^{*2} + \frac{1}{16}\,x_{32}^{*2}\right) \leq 1 \\
& u_1 \geq 0, u_2 \geq 0, \lambda_i \geq 0.0001, i = 1, \ldots, 3.
\end{array}$$

Note that we have added a small lower bound on λ; this allows us to simply represent the positive homogeneous extension. Solving yields a dual objective value of -0.2139 with multipliers $\lambda_1 = 0.0001$, $\lambda_2 = 0.0476$, $\lambda_3 = 0.0467$. Since λ_1 is at its lower bound, the corresponding term in the primal objective will not contribute to the primal objective. Hence we drop it from the model and resolve the dual. In this case, the dual objective value is -0.2144 with dual variables

$$\begin{array}{lll}
\lambda_2 = 0.0477 & x_{21}^* = 0.609 & x_{31}^* = 0.199 \\
\lambda_3 = 0.0467 & x_{22}^* = 0.039 & x_{32}^* = 0.306 \\
u_1 = 0.059 & u_2 = 0. &
\end{array}$$

Since the dual variable λ is strictly greater than 0.0001, we now have the optimal dual solution. Consequently the optimal primal value is $t = 0.214$ from (4). Further, the optimal primal solution is from (2).

$$x = {}^{(x_{21}^* t)}\Big/_{(6\lambda_2)} = 0.456$$

$$y = {}^{(x_{22}^* t)}\Big/_{(\lambda_2)} = 0.175$$

Note that, since $u_1 > 0$, constraint (5) is active while constraint (6) is inactive at optimality.

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